**Indian Statistical Institute, Bangalore**

M. Math. Second Year

Second Semester - Partial Differential Equations

Final Exam | Duration: 3 hours | Date : May 12, 2015

Answer all the questions. Max Marks: 50

1. (a) Let $P(-i\partial)$ be any Hypoelliptic Operator on $\mathbb{R}^n$. Show that $P$ satisfies the condition $C$.
   $C$ : for each constant $c_1 > 0$, there exists a constant $c_2 > 0$ such that whenever $P(\xi \ + \ i \ \eta) = 0$ and $|\eta| < c_1$, one gets $|\xi| < c_2$. [5]

   (b) Let $p$ be any polynomial satisfying condition $C$ above. Then show that

   $$\frac{p(\xi + \theta)}{p(\xi)} \to 1 \text{ as } |\xi| \to \infty, \ \xi \ \in \ \mathbb{R}^n$$

   for each $\theta$ in $\mathbb{R}^n$. [4]

   (c) Give an example of a non-elliptic but hypo elliptic operator and prove your claim. [3]

2. Show that the heat equation $u_t - u_{xx}, x$ in $\mathbb{R}$, $t$ in $\mathbb{R}$ is not analytic hypo elliptic. [7]

3. Show that [4]

   $$(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})\frac{1}{x + i\ y} = \delta_0$$

   .

4. Let $f, g : \mathbb{R} \to \mathbb{R}$ be $C^2$ functions. Let $u(x, t)$ for $t, x$ in $\mathbb{R}$ be a solution of the wave equation.

   $$u_{tt} - u_{xx} = 0$$

   with the boundary conditions.

   $$u(x, 0) = f(x)$$

   $$u_t(x, 0) = g(x)$$

   Then necessarily $u$ is given by $u(x, t) = \frac{1}{2}[f(x + t) + f(x - t)] + \frac{1}{2}\int\limits_{x-t}^{x+t} g(y)dy.$ [4]

5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be any $C^2$ function. Define the spherical mean $M(f, x, r)$ for $x$ in $\mathbb{R}^n, r \geq 0$ by

   $$M(f, x, r) = \int\limits_{|y|=1} f(x + ry)\ d\ \sigma(y)$$

Where $\sigma$ is the surface measure on $\{y\varepsilon\mathbb{R}^n : |y| = 1\} = S^{n-1}$ given by the normalization $\sigma(S^{n-1}) = 1$. Prove the Darboux equation, for $r > 0$ [7]

$$\{\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\}M(f, x, r) = \triangle_x M(f, x, r) = M(\triangle f, x, r).$$

[Hint: One form of Stokes theorem states

$$\int_G \sum \frac{\partial A_i}{\partial x_i} dx = \int_{\partial G} \sum A_i \upsilon_i d\tau$$

where $(\upsilon_1, \upsilon_2, ...\upsilon_n)$ is the outer normal derivative for $\partial G$].

6. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Let $0 < T < \infty$. Assume $u : \overline{\Omega} \times [0, T] \to \mathbb{R}$ is a nice function satisfying

$$u_t - \triangle_x u = 0 \text{ for } t \text{ in } (0, T) \text{ and } x \text{ in } \Omega.$$

Then show that $u$ assumes its maximum value on $\Omega \times \{0\}$ or on $\partial\,\Omega \times [0, T]$. [6]

7. (a) Let $f(x) = e^x, g(x) = e^x \cos(e^x) = \frac{d}{dx}[\sin(e^x)]$. Show that $g$ is a tempered distribution on $\mathbb{R}$ but $f$ is not. [1+2]

   (b) Let $P(\partial)$ be an elliptic operator on $\mathbb{R}^n$. Let degree $P = k$. Show that

$$\liminf_{|\xi|\to\infty} \frac{|P(i\xi)|}{|\xi|^k} > 0$$

   and conversely. [1+2]

8. Let a $C^2$ function $u(x, t)$ for $x$ in $\mathbb{R}, t$ in $\mathbb{R}$ satisfy

$$u_{tt} - u_{xx} = 0$$

$$u(x, 0) = f(x)$$

$$u_t(x, 0) = 0$$

$$\text{where} f\varepsilon C^\infty_{cpt}(\mathbb{R})$$

Define $w(x, t)$ for $t > 0$ by $w(x, t) = \int_{-\infty}^{\infty} ds\ u(x, s)\frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}}$ Show that $w$ satisfies $w_t - w_{xx} = 0$ with $\lim_{t\to 0} w(x, t) = f(x)$ [5]